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NOETHERIANITY OF THE SPACE OF IRREDUCIBLE REPRESENTATIONS

BY

EDWARD S. LETZTER*

Department of Mathematics, Temple University
Philadelphia, PA 19122, USA
e-mail: letzter@math.temple.edu

ABSTRACT

Let R be an associative ring with identity. We study an elementary generalization of the classical Zariski topology, applied to the set of isomorphism classes of simple left R-modules (or, more generally, simple objects in a complete abelian category). Under this topology the points are closed, and when R is left noetherian the corresponding topological space is noetherian. If R is commutative (or PI, or FBN) the corresponding topological space is naturally homeomorphic to the maximal spectrum, equipped with the Zariski topology. When R is the first Weyl algebra (in characteristic zero) we obtain a one-dimensional irreducible noetherian topological space. Comparisons with topologies induced from those on A. L. Rosenberg's spectra are briefly noted.

1. Introduction

One of the fundamental ideas in noncommutative algebraic geometry (see [10] for a recent survey) is that to each noncommutative ring R there corresponds a "noncommutative affine space." Ideally, such a space should closely reflect the representation theory of R and should follow a construction mimicking the classical commutative case. This note, then, is concerned with the "noncommutative affine space of irreducible representations of R."

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Now let R-**space** denote the set of isomorphism classes of simple R-modules. In §2 we equip R-space with the R-**topology**. It follows immediately from the definition that points (i.e., simple R-modules) are closed in the R-topology, and in §3 we prove that the R-topology is noetherian if R is a left noetherian ring. Also, if R is commutative (or PI, or FBN) then R-space, equipped with the R-topology, is naturally homeomorphic to $\max R$, equipped with the Zariski topology; see §4.

When R is the first Weyl algebra (in characteristic zero), R-space is a one-dimensional, irreducible, noetherian topological space; see (4.3–4). In particular, the R-topology can distinguish between Weyl algebras and simple Artinian rings (whose corresponding spaces are singletons).

Following [7] and [12], we actually work in a somewhat more general setting. Let A be a complete abelian category, and let A-space denote the collection of isomorphism classes of simple objects in A. Assume further that A-space is a set. In §2 we define the A-topology on A-space, and in §3 we prove that the A-topology is noetherian if A has a noetherian generator. As before, the points in A-space are closed. In §5 we provide a brief comparison of the A-topology with the topologies developed by A. L. Rosenberg in [7]. Our notion of a closed set in A-space is also related to ideas found (e.g.) in [1, 3, 4, 8]; see (2.6).

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2. A-space

2.0. PRELIMINARIES. The following notation and assumptions will remain in effect throughout this note. The reader is referred (e.g.) to [2, 6, 11] for basic background information on rings and categories.

(i) We will use R to denote an associative ring with identity. We will only use "R-module" to mean "left R-module," and the category of R-modules will be designated Mod R.

(ii) We will use A to denote a complete abelian category. Recall that an abelian category is complete if and only if it is closed under products [11, IV.8.3]. Our

primary motivating examples of complete abelian categories are Grothendieck categories [11, X.4.4] and $\operatorname{Mod} R$.

(iii) Following [12], we will refer to the objects in A as **A-modules**, and we will employ the terminology of modules when appropriate.

(iv) Let M be an A-module, and let S be a set of submodules of M. Set

$$\bigcap_{N\in S} N = \text{kernel of the natural map } M \to \prod_{N\in S} M/N.$$

Using this definition, the proof of the Schreier Refinement Theorem (cf., e.g., [2, 3.10]) can be readily adapted to series of A-modules.

(v) Let **A-space** denote the collection of isomorphism classes of simple A-modules (i.e., simple objects in A). We will assume for the remainder of this note that A-space is a set. We will use **R-space** to denote ($\operatorname{Mod} R$)-space.

(vi) For each $p \in A$-space, let N_p denote a chosen representative A-module in p. (The topological structure of A-space described below will not depend on these choices.) We will use $[N]$ to denote the isomorphism class in A of an A-module N.

2.1. Define a subset X of A-space to be an **algebraic set** if the isomorphism class of each simple subquotient of

$$\prod_{p\in X} N_p$$

is contained in X.

2.2. Remark: Suppose that R is commutative. We will see in (4.1) that the algebraic sets in R-space correspond exactly to the Zariski closed subsets of $\max R$.

2.3. THEOREM: *A-space, with the closed sets defined to be the algebraic sets, is a topological space.*

Proof: It is immediately evident that $\emptyset$ and A-space are algebraic sets. Also, it is easy to verify that the intersection of an arbitary collection of algebraic sets is an algebraic set. Now suppose that X_1 and X_2 are algebraic sets, and let $X = X_1 \cup X_2$. Then

$$\prod_{p\in X} N_p \cong \left(\prod_{p\in X_1} N_p\right) \times \left(\prod_{p\in X_2 \setminus X_1} N_p\right).$$

It now follows from the Schreier Refinement Theorem that X is an algebraic set. ∎

2.4. For convenience, we will refer to the topology defined in (2.3) as the A-**topology**.

2.5. For every (two-sided) ideal I of R, let

$$v(I) = \{p \in R\text{-space}: I \subseteq \operatorname{ann} N_p\}.$$

(i) Generalizing the well-known terminology for primitive ideals, define the **Jacobson topology** on the set R-space to be the topology in which the $v(I)$ are the closed sets. It is easy to see that the R-topology is a refinement of the Jacobson topology. When R is commutative, we will refer to the Jacobson topology as the **Zariski topology**.

(ii) Let $\operatorname{prim} R$ denote the (left) primitive spectrum of R, equipped with the usual Jacobson topology. When both R-space and $\operatorname{prim} R$ are equipped with the Jacobson topology, the map

$$\pi\colon R\text{-space} \xrightarrow{p \mapsto \operatorname{ann} N_p} \operatorname{prim} R$$

is a closed and continuous surjection. With respect to the R-topology on R-space and the Jacobson topology on $\operatorname{prim} R$, it follows from (i) that π is continuous. When R is commutative, π is a continuous bijection from R-space onto $\max R$ (equipped with the classical Zariski topology); bicontinuity in this case will follow from (4.1).

2.6. Remarks: (i) In [1], a full subcategory of an abelian category is called "closed" when it is closed under direct limits and subquotients; in [8] such subcategories are termed "weakly closed."

(ii) The notion of algebraic set presented in (2.1) is also similar to ideas developed in [3, 4].

3. Noetherianity

Recall the notation of (2.0). Assume in this section that A-space is equipped with the A-topology and that R-space is equipped with the R-topology.

3.1. (i) For each A-module M, let $S(M)$ denote the set of isomorphism classes of simple subquotients of M, and let $V(M)$ denote the closure of $S(M)$ in A-space.

(ii) For each closed subset X of A-space, let

$$M(X) = \prod_{p \in X} N_p.$$

Of course, $M(X)$ is determined up to isomorphism only by the closed set X and not by the choices of N_p for $p \in X$. It follows from the definitions, since X is closed, that $S(M(X)) = V(M(X)) = X$.

(iii) If M is an A-module we will refer to $M(V(M))$ as the **radical** of M, denoted $\sqrt{M}$. It again follows from the definitions that

$$\sqrt{\sqrt{M}} \cong \sqrt{M} \quad \text{and} \quad V(M) = V(\sqrt{M}).$$

3.2. LEMMA: *Let X be a closed subset of A-space, and let $M = M(X)$. Assume there exists an A-module E with the following property: For each $p \in X$ there is an epimorphism $g_p\colon E \to N_p$. Let $g\colon E \to M(X)$ denote the product morphism, and let F denote the image of g in $M(X)$. Then $S(F) = V(F) = X$, and $\sqrt{F} \cong M$.*

Proof: First, $S(F) \subseteq V(F) \subseteq V(M) = X$, since F is a submodule of M. On the other hand, if $p \in X$ then N_p is isomorphic to a subquotient of F. Hence $X \subseteq S(F)$. Therefore, $X = S(F)$, and $\sqrt{F} \cong M$. ∎

3.3. THEOREM: *Assume there exists a noetherian A-module E that maps epimorphically onto each simple A-module. Then A-space is a noetherian topological space.*

Proof: Let

$$X = X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$$

be a descending chain of closed subsets of A-space.

For every $p \in X$, choose an epimorphism $g_p\colon E \to N_p$. For all $i = 1, 2, 3, \ldots$, let

$$E \xrightarrow{g_i} \prod_{p \in X_i} N_p = M(X_i)$$

be the product map, and let M_i denote the image of g_i in $M(X_i)$. Observe that there is an epimorphism

$$M_i \to M_{i+1},$$

for each $i = 1, 2, \ldots$. However, since M_1 is noetherian, there exists some positive integer t for which

$$M_t \cong M_{t+1} \cong \cdots.$$

Therefore, by (3.2),

$$X_t = X_{t+1} = \cdots.$$

The theorem follows. ∎

3.4. COROLLARY: *If A possesses a noetherian generator then A-space is noetherian, and if R is left noetherian then R-space is noetherian.*

3.5. Assume that A-space is noetherian. Recalling the standard elementary notions of algebraic geometry, we see that every algebraic subset is a finite union of irreducible components. Moreover, we can define the dimension of an algebraic subset to be the supremum of the lengths of the chains of its irreducible subsets.

3.6. (My thanks to Paul Smith for the following remark; cf. [9, §7].) We can see as follows that $V(M)$ may be strictly larger than $S(M)$. Let k be a field, let $\lambda \in k$ be a nonzero nonroot of unity, and assume that $R = k\{x, y\}/\langle xy - \lambda yx\rangle$. We can regard the commutative polynomial ring $k[x]$ either as a subalgebra of R or as an R-module on which y acts trivially and x acts by left multiplication. For $\mu \in k$, set $K(\mu) = R/\langle y, x - \mu\rangle$, viewed as a 1-dimensional simple left R-module on which y acts trivially and x acts as multiplication by μ. Set

$$M = R \oslash_{k[x]} K(1) \cong R/R.(x - 1).$$

It is not hard to see that

$$S(M) = \{[K(\lambda^i)]\colon i = 0, 1, 2, \ldots\}.$$

Now let $P = \sqrt{M}$. It is not hard to verify that P contains an isomorphic copy of the left R-module $k[x]$, and it follows, for all $\mu \in k$, that $K(\mu)$ is isomorphic to a simple R-module quotient of P. Therefore, $V(M)$ strictly contains $S(M)$, and $S(M)$ is not closed in the R-topology on R-space.

4. When is the R-topology equivalent to the Jacobson topology?

If the R-topology and Jacobson topology coincide on R-space, then every primitive factor of R must have exactly one simple faithful module (up to isomorphism). In this section we first consider partial converses to this conclusion, for three specific classes of rings whose primitive factors are simple artinian: Commutative rings, PI rings, and FBN rings. We then show that the R-topology is a strict refinement of the Jacobson topology when R is the first Weyl algebra over a field of characteristic zero.

We retain the notation of the preceding sections.

4.1. PROPOSITION: *Suppose that R is a PI ring. Then the R-topology and Jacobson topology coincide on R-space. In particular, when R is commutative the Zariski topology and R-topology coincide on R-space.*

Proof: Let X be a subset of R-space closed under the R-topology, and let

$$I = \operatorname{ann} M(X) = \bigcap_{p \in X} \operatorname{ann} N_p.$$

By (2.5i), it suffices to prove that $X = v(I)$. We will assume, without loss of generality, that $I = 0$, and we will prove that X is equal to all of R-space. By Kaplansky's Theorem, there exists a positive integer n such that $R/\operatorname{ann} N$ is isomorphic to a submodule of $\bigoplus_{i=1}^{n} N$, for all simple R-modules N. Therefore, R is isomorphic to a submodule of

$$\prod_{p \in X} \left(\bigoplus_{i=1}^{n} N_p \right) \cong \bigoplus_{i=1}^{n} M(X),$$

and so every simple R-module is isomorphic to a subquotient of $M(X)$. Thus R-space is equal to X. ∎

4.2. PROPOSITION: *Suppose that R is left fully bounded left noetherian. Then the R-topology and Jacobson topology coincide on R-space.*

Proof: Let X be a subset of R-space closed under the R-topology. By (3.2), with $E = {}_R R$, there exists a cyclic R-module M such that $X = S(M)$. By (2.5i), it suffices to prove that $X = v(\operatorname{ann} M)$, and we will assume without loss of generality that $\operatorname{ann} M = 0$. Again we must show that X is equal to all of R-space. However, by Cauchon's Theorem (see, e.g., [2, 8.9]), R embeds as an R-module into a finite direct sum of copies of M. Hence every isomorphism class of simple R-modules is contained in $S(M)$, and the proposition follows. ∎

We now turn to an example where the Jacobson topology and R-topolgy are distinct.

4.3. LEMMA: *Assume that R is a domain with left Krull dimension equal to 1. Further suppose that R has infinitely many pairwise non-isomorphic simple modules. Then R-space is a one-dimensional irreducible topological space.*

Proof: Let S be any infinite collection of maximal left ideals of R for which the simple modules R/L, for $L \in S$, are pairwise non-isomorphic. Because every proper R-module factor of R has finite length, it follows that $\bigcap_{L \in S} L = 0$. Therefore, R embeds as an R-module into every direct product of infinitely many pairwise non-isomorphic simple R-modules. Consequently, R-space itself is the only infinite closed subset of R-space, and so R-space is 1-dimensional and irreducible. ∎

4.4. Example: Assume that k is a field of characteristic zero, and let R denote the first Weyl algebra, $k\{x,y\}/\langle xy-yx-1\rangle$. It is well-known that R is a simple noetherian domain of left Krull dimension 1; see, for example, [6]. Moreover, R-space is infinite (cf., e.g., [5]). It now follows from (3.4) and (4.3) that R-space is irreducible, 1-dimensional, and noetherian. Under the Jacobson topology, the only closed subsets of R-space are $\emptyset$ and R-space itself.

5. Comparisons with topologies relative to A. L. Rosenberg's spectra

In this section we assume that the reader is somewhat familiar with the terminology and notation in [7], which we will adopt. We will also continue to use the notation and conventions established in the preceding sections of this note.

5.1. Following [7, §III.1.2], A-space can be identified with a subset of Spec A (as defined in [7, §III.1.2]). Several topologies on Spec A are considered in [7], and we can therefore compare the A-topology to their induced (or, relative) topologies on A-space.

5.2. The topology τ [7, III.5.1] induces the discrete topology on A-space.

5.3. The Zariski topology on Spec(Mod R) [7, III.6.3] induces the Jacobson topology on R-space. The central topology [7, III.7.1] on Spec(Mod R) is weaker than this Zariski topology.

5.4. (i) In [7, III.7.2], the topology τ^* on Spec A is defined by declaring the set of supports of finite type objects in A to be a base. (The support, in Spec A, of an object in A is defined in [7, III.5.2].) When A is the category of modules over a commutative ring, τ^* is exactly the Zariski topology on the classical prime spectrum.

(ii) Let M be a finite type object in A. Then the set $S(M)$ of ismorphism classes of simple subquotients of M is closed under the induced τ^* topology on A-space. However, we saw in (3.6) that $S(M)$ need not be closed under the A-topology. Therefore, the A-topology and the induced τ^*-topology are distinct.

(iii) Retain the notation of (3.6). For each non-negative integer n, set

$$M(n) = R \otimes_{k[x]} K(\lambda^n) \cong R/R.(x-\lambda^n),$$

and let

$$S_n = S(M(n)) = \{[K(\lambda^i)]\colon i = n, n+1, \ldots\}.$$

As noted in (ii), each S_n is closed under the induced τ^*-topology on R-space. However,

$$S_0 \supsetneq S_1 \supsetneq S_2 \supsetneq \cdots.$$

Hence, τ^* need not be a noetherian topology on R-space when R is a noetherian ring.

(iv) Suppose that A is a Grothendieck category with a generator E of finite type. Let X be a subset of A-space closed under the A-topology. By (3.2), $X = S(F)$ for some quotient F of E, and F is of finite type. Therefore, X is closed under the induced τ^* topology. It follows, in this case, that the τ^* topology is a refinement of the A-topology.

5.5. The topology τ_S [7, III.7.3] on Spec A is defined to be the weakest topology in which the closure of a point in Spec A is equal to its set of specializations. Therefore, τ_S reduces to the Zariski topology in the commutative case, and the points in A-space are closed under the induced τ_S-topology. We do not know whether or not the topology induced by τ_S on A-space coincides, in general, with the A-topology. We also do not know whether the topology on A-space induced by τ_S is noetherian when A has a noetherian generator. A closely related question: Let M be an object in A, and further suppose that $M \in \operatorname{Spec} A$. Must $V(M) = S(M)$?

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